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Transience on the average and spontaneous symmetry breaking on graphs

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Abstract. We give a rigorous proof of the existence of spontaneous magnetization at finite temperature for classical spin models on *transient on the average* (TOA) graphs, i.e. graphs where a random walker returns to its starting point with an average probability $\bar{F} < 1$. The proof holds for models with $O(n)$ symmetry with $n \geqslant 1$, therefore including the Ising model as a particular case. This result, together with the generalized Mermin–Wagner theorem, completes the picture of phase transitions for continuous symmetry models on graphs and leads to a natural classification of general networks in terms of the two geometrical *superuniversality classes* of *recursive on the average* and *transient on the average*.

1. Introduction

The relation between spatial geometry and physical behaviour is a fundamental problem of modern theoretical physics. The influence of geometry is particularly relevant in statistical mechanics, where universality in phase transitions and critical phenomena on lattices depends strongly on large-scale topology. The most general and rigorous results concern the existence itself of spontaneous symmetry breaking. As for a discrete symmetry model, spontaneous magnetization occurs if and only if the Euclidean dimension d is >1, while for continuous symmetries the corresponding condition is $d > 2$. In the latter case, the necessary condition is proven by the Mermin–Wagner theorem [1, 2], while the sufficient condition is contained in the Frölich–Simon–Spencer result about the infrared bound [3, 4].

On a lattice this simple and exhaustive picture allows us to classify statistical models in *geometrical superuniversality classes* determined by the Euclidean dimension.

Euclidean lattices are good models for crystals and for more abstract geometrical objects, such as discretized flat spacetime. However, most real systems, such as glasses, polymers, biological systems, fractals, have irregular geometry and cannot be described by lattices. In the same way, the presence of gravitation leads to curved spacetime, which cannot be represented by lattices. To describe these more general systems, we have to switch to more general geometrical structures, namely graphs, which are networks made of points and links.

From this perspective, lattices are a class of graphs characterized by a very peculiar property: translation invariance. This implies the existence of a reciprocal lattice and of the Euclidean dimension, the latter being the number of independent generators of the translation group. The proofs of theorems [1–4] depend strongly on translation invariance and an extension to generic graphs must involve more general techniques. Recently, progress in the study of

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statistical models on infinite graphs has been achieved exploiting the algebraic approach to graph topology [5, 6].

The generalization to graphs of the Mermin–Wagner theorem [7, 8] has been a first step to understand the behaviour of spin models on inhomogeneous structures. There, the necessary condition for the existence of spontaneous magnetization for continuous symmetry spin models is given in terms of asymptotic random walks behaviour. In particular, it is proven that if $\bar{F}$, the average probability of ever returning to the starting point for a walker on the graph, is one, i.e. the graph is *recurrent on the average* (ROA), then no spontaneous magnetization occurs. This result naturally includes the lattice theorem, since Euclidean lattices in one and two dimensions turn out to be ROA.

In this work we study the case $\bar{F} < 1$, i.e. *transient on the average* (TOA) graphs and we give a proof of the existence of spontaneous magnetization at non-zero temperature for classical spin models. The result is the inversion of [7] and an extension of [3, 4], since lattices with $d > 2$ are TOA graphs. As in the lattice case, the theorem also holds for the Ising model, providing a first general result for discrete symmetry on graphs.

Since any graph can be classified either as ROA or TOA, this proof, together with theorem [7], completes the picture for the case of spontaneous breaking of continuous symmetry models on graphs. In this way we can extend to graphs the concept of geometrical superuniversality classes. The average recurrence property of random walks provides the link between the physical behaviour of the $O(n)$ model and the large-scale topology of the discrete space.

In the following section, we introduce the basic graph-theoretical techniques: the algebraic approach to graph topology, the definition of the thermodynamic limit on infinite graphs, the random walk problem. Then, in section 2 we define $O(n)$ models and their thermodynamics on infinite graphs. In section 4 we prove the existence of spontaneous magnetization for $O(n)$ models defined on a fundamental class of graphs, called pure TOA. Finally, in section 5 we extend the proof to all TOA graphs. The mathematical details of the proof will be given in the appendix.

2. Some mathematical properties of graphs

A graph $\mathcal{G}$ is a countable set V of vertices (i) connected pairwise by a set E of unoriented links $(i, j) = (j, i)$. In physical models vertices usually represent sites, where spins or fields are defined while links represent the interactions between them. If the set V is finite, $\mathcal{G}$ is called a finite graph and we will denote by N the number of vertices of $\mathcal{G}$. A subgraph $\mathcal{S}$ of $\mathcal{G}$ is a graph whose set of vertices $S \subseteq V$ and whose set of links $E' \subseteq E$.

A path in $\mathcal{G}$ is a sequence of consecutive links $\{(i, k)(k, h) \ldots (n, m)(m, j)\}$ and a graph is said to be connected, if for any two points $i, j \in V$ there is always a path joining them. In the following we will consider only connected graphs. Every connected graph $\mathcal{G}$ is endowed with an intrinsic metric generated by the chemical distance $r_{i,j}$ which is defined as the number of links in the shortest path connecting vertices i and j.

The graph topology can be described algebraically by its adjacency matrix A_{ij} given by

$$A_{ij} = \begin{cases} 1 & \text{if} \quad (i, j) \in E \\ 0 & \text{if} \quad (i, j) \notin E. \end{cases} \tag{1}$$

The Laplacian matrix Δ_{ij} is defined by

$$\Delta_{ij} = z_i \, \delta_{ij} - A_{ij} \tag{2}$$

where $z_i = \sum_j A_{ij}$, the number of nearest neighbours of i, is called the coordination number of i. Here we will consider graphs with bounded connectivity, i.e. with $\max_i z_i < \infty$.

Δ_{ij} is the generalization to graphs of the usual Laplacian on a lattice where $z_i = z\ \forall i$. If $\mathcal{G}$ is a finite graph the matrix Δ_{ij} can be consider as a symmetric operator Δ on a finite N-dimensional vector space. Δ is diagonalizable and its spectrum is real, non-negative and bounded. In particular, 0 is a simple eigenvalue of Δ_{ij} and it corresponds to the constant eigenvector. Notice that, while on a regular lattice Δ is diagonalized by the Fourier transform, this is not the case for a generic graph.

A generalization of the adjacency matrix A_{ij} is useful in the study of disordered ferromagnetic models and it is given by the coupling matrix J_{ij}:

$$J_{ij} = J_{ji} = \begin{cases} J_{ij} & \text{if} \quad A_{ij} = 1 \\ 0 & \text{if} \quad A_{ij} = 0. \end{cases} \tag{3}$$

If $\sup_{(i,j)} J_{ij} < \infty$ and $\inf_{(i,j)} J_{ij} > 0$, J_{ij} can be regarded as bounded ferromagnetic interactions between the nearest-neighbour vertices of the graphs. One can then define the generalized Laplacian

$$L_{ij} = J_i\,\delta_{ij} - J_{ij} \tag{4}$$

where $J_i = \sum_j J_{ij}$. On a finite graph, if we consider L_{ij} as a symmetric operator L on N-dimensional vector space, we have that L has the same properties of Δ (2): it is diagonalizable, its spectrum is real, positive and bounded.

Phase transitions, corresponding to singularities in the free energy of a statistical model, only occurs in the thermodynamic limit, i.e. on infinite graphs. To define a model on an infinite graph, $\mathcal{G}$ we consider the models defined on a sequence of concentric spheres in the intrinsic metric, generalizing the usual Van Hove spheres. A generalized Van Hove sphere $S_{o,r} \subset \mathcal{G}$ of centre o and radius r is the subgraph of $\mathcal{G}$ containing all $i \in \mathcal{G}$ whose distance from o is $\leqslant r$ and all the links of $\mathcal{G}$ joining them. We will call $N_{o,r}$ the number of vertices contained in $S_{o,r}$. We define the value of any physical quantity on the infinite graph $\mathcal{G}$ as the limit for $r \to \infty$ of the corresponding quantity calculated for a model on $S_{o,r}$. Given a function ϕ_i of the vertices of $\mathcal{G}$, we define its average value $\overline{\phi}$ of ϕ_i as

$$\overline{\phi} \equiv \lim_{r\to\infty} \frac{\sum_{i\in S_{o,r}} \phi_i}{N_{o,r}}. \tag{5}$$

The measure $|S|$ of a subset S of V is the average value $\overline{\chi(S)}$ of its characteristic function $\chi_i(S)$ defined by $\chi_i(S) = 1$ if $i \in S$ and $\chi_i(S) = 0$ if $i \notin S$. In an analogous way, we define the normalized trace $\overline{\mathrm{Tr}}B$ of a matrix B_{ij}:

$$\overline{\mathrm{Tr}}B \equiv \overline{b} \tag{6}$$

where $b_i \equiv B_{ii}$. It can be shown that if $\mathcal{G}$ satisfies all the conditions listed above, all the average values are independent of the centre of the spheres sequence o [11]. The necessary condition for the thermodynamic limit to be independent of boundary conditions can be expressed as a geometrical constraint on the large-scale structure of $\mathcal{G}$. Namely, one must require that

$$\lim_{r\to\infty} \frac{|\partial S_{o,r}|}{|S_{o,r}|} = 0 \tag{7}$$

where $\partial S_{o,r}$ is the boundary of the sphere S.

Occurrence of phase transitions depends on large-scale topology. On a lattice, all the relevant information about it is encoded in the space dimensionality d. On a graph, where a direct definition of dimension is lacking, a fundamental tool to characterize large-scale topology is the long-time asymptotics of random walks.

Simple discrete time random walks are defined on a graph $\mathcal{G}$ by the jumping probabilities p_{ij} between nearest-neighbour sites i and j, which are expressed in terms of the adjacency matrix:

$$p_{ij} = \frac{A_{ij}}{z_i} = (Z^{-1}A)_{ij} \tag{8}$$

where $Z_{ij} = z_i\,\delta_{ij}$. From (8) the probability of reaching in t steps site j starting from i is given by

$$P_{ij}(t) = (p^t)_{ij}. \tag{9}$$

Graph topology deeply affects a conditional probability related to P_{ij}, i.e. the probability $F_{ii}(t)$ of returning to the starting point i for the first time after t steps. The relation between the two probabilities can be expressed simply in terms of their generating functions $\tilde{P}_i(\lambda)$ and $\tilde{F}_i(\lambda)$ defined by

$$\tilde{P}_i(\lambda) = \sum_{t=0}^{\infty} \lambda^t P_{ii}(t) \qquad \tilde{F}_i(\lambda) = \sum_{t=0}^{\infty} \lambda^t F_{ii}(t). \tag{10}$$

By standard Markov chains properties one obtains

$$\tilde{F}_i(\lambda) = \frac{\tilde{P}_i(\lambda) - 1}{\tilde{P}_i(\lambda)}. \tag{11}$$

In particular, $\tilde{F}_i(1)$ is the probability of ever returning to the starting point i and it only depends on graph topology. Its average value over all starting sites $\bar{F} \equiv \lim_{\lambda\to 1^-} \overline{\tilde{F}(\lambda)}$ classifies all graphs in two families, which we will call *recurrent on the average* and *transient on the average*. A graph is said to be ROA if $\bar{F} = 1$ and TOA in the opposite case, i.e. if $\bar{F} < 1$.

When dealing with thermodynamic properties one is forced to consider only the average value of F_i over all starting points of the graphs. Indeed, the standard mathematical classification of infinite graphs into (*locally*) transient and recursive, based on the condition $F_i = 1$ (for at least one i and therefore for all i) does not coincide with that based on average values: locally transient graphs can be recurrent on the average [7, 9, 10].

The TOA family must be further divided into two subfamilies: pure TOA and mixed TOA. A TOA graph is pure if the average value of $\tilde{F}_i(1)$ is < 1 on every positive measure subset $S \subset V$:

$$\lim_{\lambda\to 1^-} \frac{\overline{\chi(S)\tilde{F}(\lambda)}}{|S|} < 1 \qquad \forall S \subset V \quad |S| > 0. \tag{12}$$

When condition (12) is not satisfied, i.e. there exists a subset R of V where the average value of $\tilde{F}_{ii}$ is 1, the graph is said to be a mixed TOA. In this case $\mathcal{G}$ can be decomposed into a pure TOA subgraph $\mathcal{S}$ and a ROA subgraph $\bar{\mathcal{S}}$ by cutting a zero measure set of links $\partial(\mathcal{S}, \bar{\mathcal{S}}) \equiv \{(i, j) \in E\,|i \in \mathcal{S} \wedge j \in \bar{\mathcal{S}}\}$ [11].

This classification has a deep geometrical origin and it is left invariant by a redefinition of jumping probabilities in terms of the bounded interaction matrix J_{ij} [9], i.e. by considering a rescaled set of p_{ij} obtained by replacing A_{ij} with J_{ij} in (8):

$$p_{ij} = \frac{J_{ij}}{J_i}. \tag{13}$$

On a pure TOA graph, the following fundamental properties holds, which we will call *infrared boundedness* [11]:

$$\lim_{\mu\to 0} \overline{\mathrm{Tr}}(L + \mu)^{-1} = v < \infty \tag{14}$$

where $\mu > 0$, while on ROA and mixed TOA one has

$$\lim_{\mu\to 0} \overline{\mathrm{Tr}}(L+\mu)^{-1} = \infty. \tag{15}$$

Our result on the magnetization bound is based on the infrared boundedness property (14).

3. Classical spin models on graphs

Classical spin models are classified according to the symmetry of the Hamiltonian. The typical symmetry can be described by the $O(n)$ group, for integer $n \geqslant 1$. The simplest Hamiltonian satisfying this symmetry can be written as

$$\mathcal{H} = -\tfrac{1}{2}\sum_{ij} J_{ij}\vec{\sigma}_i\cdot\vec{\sigma}_j - \vec{h}\sum_i \vec{\sigma}_i \tag{16}$$

where J_{ij} is a bounded ferromagnetic interaction matrix on the graph $\mathcal{G}$ and $\vec{\sigma}_i \equiv (\sigma_i^1,\ldots,\sigma_i^n)$ are n-dimensional real vectors defined on every vertex with the constraint

$$\vec{\sigma}_i^2 = 1 \quad \forall i. \tag{17}$$

When $n = 1$, $\mathcal{H}$ describes an Ising model, with a discrete Z_2 symmetry, while for $n \geqslant 2$ $\mathcal{H}$ represents a model with continuous symmetry. The external magnetic field $\vec{h}$ is chosen to be along the 1 direction: $\vec{h} = (h, 0, \ldots, 0)$; $h \geqslant 0$.

By (17) $\mathcal{H}$ can be rewritten in the following form which differs from (16) only by an additive constant:

$$\mathcal{H} = \tfrac{1}{4}\sum_{ij} J_{ij}(\vec{\sigma}_i - \vec{\sigma}_j)^2 - \vec{h}\sum_i \vec{\sigma}_i = \tfrac{1}{2}\sum_{ij} L_{ij}\vec{\sigma}_i\vec{\sigma}_j - \vec{h}\sum_i \vec{\sigma}_i \tag{18}$$

where L is the Laplacian operator (4).

In the canonical Boltzmann ensemble, each configuration $\{\vec{\sigma}_i\}$ has the statistical weight $\exp[-\beta\mathcal{H}(\vec{\sigma}_i,\vec{h})]$ where $\beta = 1/kT$. The free energy f of the model in the thermodynamic limit is defined by

$$f \equiv -\lim_{r\to\infty}\frac{1}{\beta N_{o,r}}\ln\int\prod_{i\in S_{o,r}} \mathrm{d}\vec{\sigma}_i\,\delta(\vec{\sigma}_i^2 - 1)\,\mathrm{e}^{-\beta\mathcal{H}_r(\{\vec{\sigma}_i\},h)} \tag{19}$$

where $\mathcal{H}_r$ is the restriction of the Hamiltonian (18) to the subgraph $S_{o,r}$.

The order parameter is the average magnetization along the $\vec{h}$ direction:

$$M(\beta,h) \equiv \lim_{r\to\infty}\frac{1}{N_{o,r}}\sum_{i\in S_{o,r}}\langle\sigma_i^1\rangle \equiv \overline{\langle\sigma^1\rangle} \tag{20}$$

where the brackets denote the thermal average. One then obtains

$$M(\beta,h) = \lim_{r\to\infty}\frac{1}{N_{o,r}}\frac{\int\prod_{i\in S_{o,r}}\mathrm{d}\vec{\sigma}_i\,\delta(\vec{\sigma}_i^2-1)\Big(\sum_{i\in S_{o,r}}\sigma_i^1\Big)\,\mathrm{e}^{-\beta\mathcal{H}_r}}{\int\prod_{i\in S_{o,r}}\mathrm{d}\vec{\sigma}_i\,\delta(\vec{\sigma}_i^2-1)\,\mathrm{e}^{-\beta\mathcal{H}_r}}. \tag{21}$$

On graphs the order parameter must be an average quantity. Local definitions as $\langle\sigma_i\rangle$ or $\lim_{r_{i,j}\to\infty}\langle\sigma_i\cdot\sigma_j\rangle$, which on lattices are equivalent to (20), on inhomogeneous structures can give ambiguous results in different sites and in general they are not equivalent to (20), which is the only definition that describes the global behaviour of the model.

In the next sections, we will show that on TOA graphs for small enough temperature we have

$$\lim_{h\to 0} M(\beta,h) \geqslant c(\beta) > 0. \tag{22}$$

The bound on $M(h,\beta)$ (equation (22)), proves the existence of spontaneous symmetry breaking in $O(n)$ models.

4. The pure case

Let us consider first a model defined on a pure TOA graph. The bound on the magnetization (22) is obtained according to the following steps:

(a) we introduce for the constraint (17) an integral representation using new variables, the Lagrange multipliers α_i. Substituting the integral representation in the expressions of the magnetization (21) and in the identity

$$1 = \frac{1}{N_{o,r}} \frac{\int \prod_{i\in S_{or}} \mathrm{d}\vec{\sigma}_i \sum_{j\in S_{or}} \vec{\sigma}_j^2 \,\mathrm{e}^{-\beta\mathcal{H}_r} \prod_{k\in S_{or}} \delta(\vec{\sigma}_k^2 - 1)}{\int \prod_{i\in S_{or}} \mathrm{d}\vec{\sigma}_i \,\mathrm{e}^{-\beta\mathcal{H}_r} \prod_{k\in S_{or}} \delta(\vec{\sigma}_k^2 - 1)} \tag{23}$$

we can perform the Gaussian integral with respect to the spin variables $\vec{\sigma}_i$;

(b) we determine the asymptotic behaviour of the integrals over α_i for $\beta \to \infty$ by a saddle-point technique.

(c) we prove two basic inequalities for the inverse of the Laplacian matrix, which will be used to compare the value of the magnetization with the expression given by (23).

Let us start with point (a) of our proof, writing the spherical constraint with the complex integral representation of the delta function:

$$\delta(\vec{\sigma}_i^2 - 1) = \frac{\mathrm{e}^{\epsilon/2}}{2\pi} \int \mathrm{d}\alpha_i \exp(-\mathrm{i}\alpha_i(\vec{\sigma}_i^2 - 1)/2 - \epsilon\vec{\sigma}_i^2/2) \tag{24}$$

where ϵ is a real arbitrary constant. We will chose $\epsilon = h\beta$. Substituting expression (24) for $\delta(\vec{\sigma}_i^2 - 1)$ in (21) and in (23) evaluated on the finite subgraph $S_{o,r}$, we obtain

$$M_r(h) = \frac{1}{N_{o,r}} \frac{\int \prod_{k\in S_{o,r}} \mathrm{d}\vec{\sigma}_k \int_{-\infty}^{+\infty} \prod_{j\in S_{o,r}} \mathrm{d}\alpha_j \, g_{\beta,h}(\sigma,\alpha) \left(\sum_{i\in S_{o,r}} \sigma_i^1\right)}{\int \prod_{k\in S_{o,r}} \mathrm{d}\vec{\sigma}_k \int_{-\infty}^{+\infty} \prod_{j\in S_{o,r}} \mathrm{d}\alpha_j \, g_{\beta,h}(\sigma,\alpha)}$$

$$1 = \frac{1}{N_{o,r}} \frac{\int \prod_{k\in S_{o,r}} \mathrm{d}\vec{\sigma}_k \int_{-\infty}^{+\infty} \prod_{j\in S_{o,r}} \mathrm{d}\alpha_j \, g_{\beta,h}(\sigma,\alpha) \left(\sum_{i\in S_{o,r}} \vec{\sigma}_i^2\right)}{\int \prod_{k\in S_{o,r}} \mathrm{d}\vec{\sigma}_k \int_{-\infty}^{+\infty} \prod_{j\in S_{o,r}} \mathrm{d}\alpha_j \, g_{\beta,h}(\sigma,\alpha)}$$

where

$$g_{\beta,h}(\sigma,\alpha) = \exp\left(-\beta\left(\tfrac{1}{2}\sum_{ij} L_{ij}\vec{\sigma}_i\vec{\sigma}_j - h\sum_i \sigma_i^1\right) - \tfrac{1}{2}\sum_i(\mathrm{i}\alpha_i(\vec{\sigma}_i^2 - 1) - h\beta\vec{\sigma}_i^2)\right).$$

Rescaling in both integral α_i by $\beta\alpha_i$ and $\vec{\sigma}_i$ by $\vec{\sigma}_i/\sqrt{\beta}$, we can perform the Gaussian integration on the variables $\vec{\sigma}_i$ obtaining

$$M_r(h) = \frac{1}{Z}\int \mathrm{d}\mu_{\beta,h}(\alpha)\, \frac{h}{N_{o,r}} \sum_{kj} (L + H + \mathrm{i}\alpha)_{kj}^{-1} \tag{25}$$

$$1 = \frac{1}{Z}\int \mathrm{d}\mu_{\beta,h}(\alpha)\, \frac{n}{\beta N_{o,r}} \mathrm{Tr}(L + H + \mathrm{i}\alpha)^{-1} + \frac{h^2}{N_{o,r}} \sum_{ij} (L + H + \mathrm{i}\alpha)_{ij}^{-2} \tag{26}$$

where $\mathrm{d}\mu_{\beta,h}(\alpha)$ is a measure on the space of the Lagrange multipliers α_i given by:

$$\mathrm{d}\mu_{\beta,h}(\alpha) = \prod_{i\in S_{o,r}} \mathrm{d}\alpha_i \exp\Bigg(-\tfrac{1}{2} n\, \mathrm{Tr}(\ln(L + H + \mathrm{i}\alpha)) \\ + \tfrac{1}{2}\beta\left(\mathrm{i}\sum_i \alpha_i + h^2 \sum_{ij} (L + H + \mathrm{i}\alpha)_{ij}^{-1}\right)\Bigg) \tag{27}$$

and

$$Z = \int \mathrm{d}\mu_{\beta,h}(\alpha_i) \tag{28}$$

with $H_{ij} = h\delta_{ij}$ and $\alpha_{ij} = \alpha_i\delta_{ij}$. Expressions (25) and (26) are the statistical averages of the quantities $h\sum_{kj}(L+H+\mathrm{i}\alpha)^{-1}_{kj}$ and $n/\beta\,\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1}+h^2\sum_{ij}(L+H+\mathrm{i}\alpha)^{-2}_{ij}$ with respect to the measure (27). Notice that the order of the symmetry group n becomes a parameter of the integration.

Let us consider our model in the very low-temperature region, that is point (b) of our proof. When $\beta\to\infty$, integrals (25) and (26) can be studied by a saddle-point technique. In particular, we have that the leading asymptotic behaviour is given by the $\bar{\alpha}_i$ for which is stationary the quantity: $\mathrm{i}\sum_i\bar{\alpha}_i + h^2\sum_{ij}(L+H+\mathrm{i}\bar{\alpha})^{-1}_{ij}$. Then the $\bar{\alpha}_i$ satisfy the equations:

$$\frac{\partial}{\partial\bar{\alpha}_i}\left[\mathrm{i}\sum_k\bar{\alpha}_k + h^2\sum_{kj}(L+H+\mathrm{i}\bar{\alpha})^{-1}_{kj}\right] = 0 \quad \forall i \tag{29}$$

where $\bar{\alpha}_{ij} = \bar{\alpha}_i\delta_{ij}$. In appendix A we will show that conditions (29) are satisfied for all values of $h\geqslant 0$ if and only if $\bar{\alpha}_i = 0\ \forall\, i$. Now we can separate integrals (25) and (26) into two parts: one is given by the integration in a small region Γ around the stationary point $\bar{\alpha}_i$ (the leading contribution) and the other is the integration over the complement of Γ. The latter vanishes for $\beta\to\infty$. We obtain

$$M_r(h) = \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\,\frac{h}{N_{o,r}}\sum_{kj}(L+H+\mathrm{i}\alpha)^{-1}_{kj} + \mathrm{o}(1/\beta) \tag{30}$$

$$1 = \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\left[\frac{n}{\beta N_{o,r}}\,\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1} + \frac{h^2}{N_{o,r}}\sum_{ij}(L+H+\mathrm{i}\alpha)^{-2}_{ij}\right] + \mathrm{o}(1/\beta) \tag{31}$$

where Γ is the region around the saddle point $\bar{\alpha}_i$ in which $\mathrm{Re}(\exp(\mathrm{i}S_{\beta h}(\alpha))) > 0$ and $Z' = \int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]$. Here we have exploited the fact that, since the measure $\mathrm{d}\mu_{\beta,h}(\alpha)$ is real and positive at the stationary point $\bar{\alpha}_i$, the imaginary part of $\mathrm{d}\mu_{\beta,h}(\alpha)$ gives a subleading contribution to integrals (30) and (31). Now $M(h)$ and 1 are real quantities so we have

$$M_r(h) = \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\,\frac{h}{N_{o,r}}\,\mathrm{Re}\left[\sum_{kj}(L+H+\mathrm{i}\alpha)^{-1}_{kj}\right] + \mathrm{o}(1/\beta) \tag{32}$$

$$1 = \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\,\mathrm{Re}\left[\frac{n}{\beta N_{o,r}}\,\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1} + \frac{h^2}{N_{o,r}}\sum_{ij}(L+H+\mathrm{i}\alpha)^{-2}_{ij}\right] + \mathrm{o}(1/\beta). \tag{33}$$

As for (c) we introduce on a finite graph the following inequalities which will be proven in appendix B exploiting the boundedness and the non-negativity of the Laplacian operator:

$$\mathrm{Re}\left[h\sum_{ij}(L+H+\mathrm{i}\alpha)^{-1}_{ij}\right] \geqslant \mathrm{Re}\left[h^2\sum_{ij}(L+H+\mathrm{i}\alpha)^{-2}_{ij}\right] \tag{34}$$

$$\mathrm{Re}[\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1}] \leqslant \mathrm{Tr}(L+H)^{-1}. \tag{35}$$

In (32) and (33) the measure is positive definite and therefore we can use (34) to compare integrals (32) and (33), obtaining

$$\begin{aligned} M_r(h) &\geqslant \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\,\mathrm{Re}\left[\frac{h^2}{N_{o,r}}\sum_{ij}(L+H+\mathrm{i}\alpha)^{-2}_{ij}\right] + \mathrm{o}(1/\beta) \\ &\geqslant 1 - \mathrm{o}(1/\beta) - \frac{1}{Z'}\int_\Gamma \mathrm{Re}[\mathrm{d}\mu_{\beta,h}(\alpha)]\,\mathrm{Re}\left[\frac{n}{\beta N_{o,r}}\,\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1}\right]. \end{aligned} \tag{36}$$

Using (35), we obtain for $M_r(h)$ the following inequality:

$$M_r(h) \geqslant 1 - \mathrm{o}(1/\beta) - \frac{1}{\beta N_{o,r}} \operatorname{Tr}(L + H)^{-1}. \qquad (37)$$

Inequality (37) holds on the finite subgraph $S_{o,r}$ and at this step we can take the thermodynamic limit, letting $r \to \infty$:

$$\lim_{r\to\infty} M_r(h) \geqslant 1 - \mathrm{o}(1/\beta) - \frac{1}{\beta}\overline{\operatorname{Tr}}(L + H)^{-1}. \qquad (38)$$

Finally, we consider the limit $h \to 0$ and, exploiting properties (14) of pure TOA graphs, we obtain

$$\lim_{h\to 0} M(h) \geqslant 1 - \frac{v}{\beta} - \mathrm{o}(1/\beta). \qquad (39)$$

This inequality gives the lower magnetization bound for pure TOA graphs.

5. The mixed TOA case

Let us now consider an $O(n)$ model defined on a mixed TOA graph. In this case the graph $\mathcal{G}$ can be decomposed into a pure TOA subgraph $\mathcal{S}$, therefore satisfying the infrared boundedness condition (14), and its complement $\bar{\mathcal{S}}$, which is a ROA graph. Exploiting the property $|\partial(\mathcal{S}, \bar{\mathcal{S}})| = 0$, in appendix C is proven for the free energy per site f that

$$f = |\mathcal{S}| f_{\mathcal{S}} + |\bar{\mathcal{S}}| f_{\bar{\mathcal{S}}} \qquad (40)$$

where $f_{\mathcal{S}}$ is the free energy of the $O(n)$ model defined on the graph $\mathcal{S}$ by the interaction matrix:

$$J_{ij}^{S} = J_{ji}^{S} = \begin{cases} J_{ij} & \text{if } (i, j) \in S \text{ and } i, j \in S \\ 0 & \text{otherwise} \end{cases} \qquad (41)$$

and $f_{\bar{\mathcal{S}}}$ is the analogous quantity defined on $\bar{\mathcal{S}}$. Equation (40) is very general and it is fundamental to show that all the thermodynamic properties of models defined on subgraphs separated by zero measure boundary are completely independent. From (40) we obtain for the magnetization

$$M(h) = -\frac{\partial f}{\partial h} = -|\mathcal{S}|\frac{\partial f_{\mathcal{S}}}{\partial h} - |\bar{\mathcal{S}}|\frac{\partial f_{\bar{\mathcal{S}}}}{\partial h} = |\mathcal{S}| M_{\mathcal{S}} + |\bar{\mathcal{S}}| M_{\bar{\mathcal{S}}} \qquad (42)$$

where $M_{\mathcal{S}}$ and $M_{\bar{\mathcal{S}}}$ are the magnetizations of the models defined on $\mathcal{S}$ and $\bar{\mathcal{S}}$. From Griffith's inequalities [12] we have that $M_{\bar{\mathcal{S}}} \geqslant 0$ and then

$$M(h) \geqslant |\mathcal{S}| M_{\mathcal{S}}. \qquad (43)$$

Now, $M_{\mathcal{S}}$ is the magnetization of an $O(n)$ model defined on a pure TOA graph and therefore the inequality (39) holds. We then obtain

$$\lim_{h\to 0} M(h) \geqslant |\mathcal{S}| - \frac{v'}{\beta} - \mathrm{o}(1/\beta). \qquad (44)$$

Since $\mathcal{S}$ is a positive measure subgraph, equation (44) proves the lower bound on $M(h)$ for a generic TOA graph.

Appendix A. The saddle-point condition

Let us now prove that the saddle-point conditions (29) are satisfied for each i only if $\alpha_k = 0\ \forall k$. Here $i \in S_{o,r}$ and L_{ij} is the restriction of the Laplacian to the Van Hove sphere $S_{o,r}$.

Taking in (29) the derivative with respect α_i, we obtain

$$\mathrm{i} - \mathrm{i}h^2 \sum_{klmj} (L + H + \mathrm{i}\bar{\alpha})^{-1}_{kl} \delta_{li} \delta_{m,i} (L + H + \mathrm{i}\bar{\alpha})^{-1}_{mj} = 0 \tag{A1}$$

where we used the fact that $\partial(\bar{\alpha}_{lm})/\partial\bar{\alpha}_i = \delta_{li}\delta_{mi}$. Then we have

$$1 = h^2 \sum_{kj} (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ki} (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ij} = \left(h \sum_k (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ki} \right)^2. \tag{A2}$$

Taking the square roots:

$$\pm 1 = h \sum_k (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ki} \tag{A3}$$

since only the choice of the sign + in each equation gives a solution with real α_i, we obtain

$$1 = h \sum_k (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ki}. \tag{A4}$$

Now (A4) must hold for all $i \in S_{o,r}$ so we have that condition (A4) is equivalent to

$$\sum_i (L + \mathrm{i}\bar{\alpha} + H)_{ij} = h \sum_{ki} (L + H + \mathrm{i}\bar{\alpha})^{-1}_{ki} (L + H + \mathrm{i}\bar{\alpha})_{ij} \qquad \forall j \in S_{o,r}. \tag{A5}$$

Since $\sum_i L_{ij} = 0$, $\sum_i \bar{\alpha}_{ij} = \bar{\alpha}_j$ and $\sum_i H_{ij} = h$, we have

$$h + \mathrm{i}\bar{\alpha}_j = h \sum_k \delta_{kj} = h \qquad \forall j \in S_{o,r}. \tag{A6}$$

Therefore, equation (29) is satisfied for all i if and only if $\bar{\alpha}_j = 0$ for all j. This proves the saddle-point condition.

Appendix B. Inequalities

Here, we give the full proof of inequalities (34) and (35). More generally we will show that the following inequalities hold for any finite graphs, with $H_{ij} \equiv h\,\delta_{ij}$, $h > 0$, $\alpha \equiv \alpha_i\,\delta_{ij}$, $\alpha_i \in \mathbb{R}$ and $\phi_i \in \mathbb{R}\ \forall i$:

$$0 \leqslant \mathrm{Re}\left(\sum_{ij} \phi_i (L + H + \mathrm{i}\alpha)^{-1}_{ij} \phi_j \right) \leqslant \sum_{ij} \phi_i (L + H)^{-1}_{ij} \phi_j \leqslant h^{-1} \sum_i \phi_i^2 \tag{B7}$$

$$\mathrm{Im}\left(\sum_{ij} \phi_i (L + H + \mathrm{i}\alpha)^{-1}_{ij} \phi_j \right) \leqslant \tfrac{1}{2} \sum_{ij} \phi_i (L + H)^{-1}_{ij} \phi_j \tag{B8}$$

$$\mathrm{Re}\left(h^2 \sum_{ij} (L + H + \mathrm{i}\alpha)^{-2}_{ij} \phi_i \phi_j \right) \leqslant \mathrm{Re}\left(h \sum_{ij} (L + H + \mathrm{i}\alpha)^{-1}_{ij} \phi_i \phi_j \right) \tag{B9}$$

$$0 \leqslant h \sum_{ikj} \phi_i (L + H - \mathrm{i}\alpha)^{-1}_{ik} (L + H + \mathrm{i}\alpha)^{-1}_{kj} \phi_j \leqslant \mathrm{Re}\left(\sum_{ij} \phi_i (L + H + \mathrm{i}\alpha)^{-1}_{ij} \phi_j \right). \tag{B10}$$

When evaluating (B9) for $\phi_i = 1\ \forall i$ we obtain inequality (34), while by considering (B7) with $\phi_i = 1$ for $i = k$ and $\phi_i = 0$ for $i \neq k$, we have

$$\mathrm{Re}(L + H + \mathrm{i}\alpha)^{-1}_{kk} \leqslant (L + H)^{-1}_{kk}.$$

Now we can sum this expression over all possible k obtaining inequality (35):

$$\mathrm{Re}(\mathrm{Tr}(L+H+\mathrm{i}\alpha)^{-1}) = \sum_k \mathrm{Re}(L+H+\mathrm{i}\alpha)^{-1}_{kk} \leqslant \sum_k (L+H)^{-1}_{kk} = \mathrm{Tr}(L+H)^{-1}.$$

In order to prove inequalities (B7)–(B10) we first introduce the N-dimensional space given by vectors $\langle\phi| = (\phi_1,\ldots,\phi_N)$ and $|\phi\rangle = (\phi_1,\ldots,\phi_N)^t$ with scalar product

$$\langle\phi|\psi\rangle = \sum_i \phi_i\psi_i. \tag{B11}$$

The matrices L, A, h and α are operator on this space. Since the matrix L is diagonalizable by a real transformation, and its spectrum l is positive: $0 \leqslant l \leqslant l_{\max}$, it can be proven [13] that it exist a real operator B which satisfies the following:

$$B^t(L+H)B = I \qquad B^t\alpha B = c \tag{B12}$$

where c is a real diagonal operator ($c_{ij} = c_i\delta_{ij}$) and I is the identity operator. Furthermore, we have

$$(L+H+\mathrm{i}\alpha)^{-1} = B(1+\mathrm{i}c)^{-1}B^t \qquad (L+H)^{-1} = BB^t \tag{B13}$$

$$\|B^tB\| = \frac{1}{h} \tag{B14}$$

where

$$\|B^tB\| = \sup_\phi \frac{\langle\phi|B^tB|\phi\rangle}{\langle\phi|\phi\rangle}.$$

Properties (B13) directly follow from (B12). Therefore, it is easy to obtain the exact expression for B:

$$B = TAT'$$

where T is the orthogonal transformation that diagonalize L; A is the transformation $(1/\sqrt{l_k+h})\delta_{km}$ where l_k is the eigenvalue of L relative to the eigenvector k; finally T' is the orthogonal operator that diagonalize the symmetric matrix $AT^t\alpha TA$. B is not an orthogonal transformation but its norm can be computed, proving (B14):

$$\|B^tB\| = \sup_\phi \frac{\langle\phi|B^tB|\phi\rangle}{\langle\phi|\phi\rangle} = \sup_\phi \frac{\langle\phi|A^2|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{1}{h}.$$

Let us consider the product $\langle\phi|(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle$. Exploiting property (B13) and inequality $(1+c_i^2)^{-1} \leqslant 1$ we have

$$\begin{aligned}
\mathrm{Re}\langle|\phi(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle &= \mathrm{Re}\langle\phi|B(1+\mathrm{i}c)^{-1}B^t|\phi\rangle \\
&= \langle\phi|(B\,\mathrm{Re}(1+\mathrm{i}c)^{-1}B^t)|\phi\rangle \\
&= \langle\phi|B(1+c^2)^{-1}B^t|\phi\rangle \\
&\leqslant \langle\phi|BB^t|\phi\rangle.
\end{aligned}$$

Now since $0 \leqslant \langle\phi|B(1+c^2)^{-1}B^t|\phi\rangle$, $\langle\phi|(L+H)^{-1}|\phi\rangle = \langle\phi|BB^t|\phi\rangle$ (equation (B13)) and $\langle\phi|BB^t|\phi\rangle \leqslant h^{-1}\langle\phi|\phi\rangle$ (equation (B14)), we obtain

$$0 \leqslant \mathrm{Re}\langle|\phi(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle \leqslant \langle\phi|(L+H)^{-1}|\phi\rangle \leqslant h^{-1}\langle\phi|\phi\rangle$$

which corresponds to (B7). In an analogous way, for the imaginary part we obtain (B8):

$$\begin{aligned}|\operatorname{Im}\langle|\phi(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle| &= |\langle\phi|Bc(1+c^2)^{-1}B^t|\phi\rangle| \\ &\leqslant \langle\phi|B|c|(1+c^2)^{-1}B^t|\phi\rangle \\ &\leqslant \tfrac{1}{2}\langle\phi|BB^t|\phi\rangle \\ &\leqslant \tfrac{1}{2}\langle\phi|(L+H)^{-1}|\phi\rangle\end{aligned}$$

where we used the inequality $|c_i|(1+c_i^2)^{-1} \leqslant \frac{1}{2}$. For the proof of (B9) we have

$$\begin{aligned}\operatorname{Re}\langle\phi|h^2(L+H+\mathrm{i}\alpha)^{-2}|\phi\rangle &= \operatorname{Re} h^2\langle\phi|B(1+\mathrm{i}c)^{-1}B^tB(1+\mathrm{i}c)^{-1}B^t|\phi\rangle \\ &= h^2\langle\phi|B(\operatorname{Re}(1+\mathrm{i}c)^{-1})B^tB(\operatorname{Re}(1+\mathrm{i}c)^{-1})B^t|\phi\rangle \\ &\quad -h^2\langle\phi|B(\operatorname{Im}(1+\mathrm{i}c)^{-1})B^tB(\operatorname{Im}(1+\mathrm{i}c)^{-1})B^t|\phi\rangle \\ &\leqslant h^2\langle\phi|B(\operatorname{Re}(1+\mathrm{i}c)^{-1})B^tB(\operatorname{Re}(1+\mathrm{i}c)^{-1})B^t|\phi\rangle \\ &\leqslant h\langle\phi|B(\operatorname{Re}(1+\mathrm{i}c)^{-1})(\operatorname{Re}(1+\mathrm{i}c)^{-1})B^t|\phi\rangle \\ &\leqslant h\langle\phi|B(1+c^2)^{-2}B^t|\phi\rangle \\ &\leqslant h\langle\phi|B(1+c^2)^{-1}B^t|\phi\rangle = h\langle\phi|(L+H)^{-1}|\phi\rangle\end{aligned}$$

where we used properties (B13), (B14) and $(1+c_i^2)^{-2} \leqslant (1+c_i^2)^{-1}$. Finally, to obtain (B10) we have

$$\begin{aligned}0 &\leqslant h\langle\phi|(L+H-\mathrm{i}\alpha)^{-1}(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle = h\langle\phi|B(1-\mathrm{i}c)^{-1}B^tB(1+\mathrm{i}c)^{-1}B^t|\phi\rangle \\ &\leqslant \langle\phi|B(1-\mathrm{i}c)^{-1}(1+\mathrm{i}c)^{-1}B^t|\phi\rangle \\ &\leqslant \langle\phi|B(1+c^2)^{-1}B^t|\phi\rangle = \operatorname{Re}\langle\phi|(L+H+\mathrm{i}\alpha)^{-1}|\phi\rangle\end{aligned}$$

and this completes the proof of inequalities (B7)–(B10).

Appendix C. Separability and the additivity of the free energy

In this last appendix we will prove the property (40) for the free energy of an $O(n)$ model when the complementary subgraphs $\mathcal{S}$ and $\bar{\mathcal{S}}$ are connected by a zero-measure border $\partial(\mathcal{S},\bar{\mathcal{S}})$.

From the definition of the free energy (19) we have that

$$\begin{aligned}f_{\mathcal{G}} - |\mathcal{S}|f_{\mathcal{S}} - |\bar{\mathcal{S}}|f_{\bar{\mathcal{S}}} &= \lim_{r\to\infty}\frac{1}{\beta N_{o,r}} \\ &\quad\times \ln\frac{\int\prod_{i\in S_{o,r}}\mathrm{d}\vec{\sigma}_i\,\delta(\vec{\sigma}_i^2-1)\exp\left(-\beta(\mathcal{H}_{r,\mathcal{S}}+\mathcal{H}_{r,\bar{\mathcal{S}}}+\sum_{(i,j)\in S_{o,r}\cap\partial(\mathcal{S},\bar{\mathcal{S}})}J_{ij}\sigma_i\cdot\sigma_j)\right)}{\int\prod_{i\in S_{o,r}}\mathrm{d}\vec{\sigma}_i\,\delta(\vec{\sigma}_i^2-1)\exp(-\beta(\mathcal{H}_{r,\mathcal{S}}+\mathcal{H}_{r,\bar{\mathcal{S}}}))} \\ &= \lim_{r\to\infty}\frac{1}{\beta N_{o,r}}\ln\left\langle\exp\left(-\beta\sum_{(i,j)\in S_{o,r}\cap\partial(\mathcal{S},\bar{\mathcal{S}})}J_{ij}\sigma_i\cdot\sigma_j\right)\right\rangle'\end{aligned} \tag{C15}$$

where $\mathcal{H}_{r,\mathcal{S}}$ and $\mathcal{H}_{r,\bar{\mathcal{S}}}$ are the restrictions of $\mathcal{H}$ to the intersections of $S_{o,r}$ with $\mathcal{S}$ and $\bar{\mathcal{S}}$ and $\langle\,\rangle'$ is an average taken with respect the statistical weight $\exp(-\beta(\mathcal{H}_{r,\mathcal{S}}+\mathcal{H}_{r,\bar{\mathcal{S}}}))$. Since the symmetry group $O(n)$ is compact ($|\sigma_i\cdot\sigma_j)|\leqslant 1$) we have that

$$-\sup_{(i,j)}J_{ij}|\partial(\mathcal{S},\bar{\mathcal{S}})|_r \leqslant \sum_{(i,j)\in S_{o,r}\cap\partial(\mathcal{S},\bar{\mathcal{S}})}J_{ij}\sigma_i\cdot\sigma_j \leqslant \sup_{(i,j)}J_{ij}|\partial(\mathcal{S},\bar{\mathcal{S}})|_r \tag{C16}$$

where $|\partial(\mathcal{S},\bar{\mathcal{S}})|_r$ is the number of links of $\partial(\mathcal{S},\bar{\mathcal{S}})$ which belong to the Van Hove sphere $S_{o,r}$. Now with (C15) and (C16) we obtain

$$\lim_{r\to\infty}\frac{1}{\beta N_{o,r}}\ln\left(\mathrm{e}^{-\beta\sup_{(i,j)}J_{ij}|\partial(\mathcal{S},\bar{\mathcal{S}})|_r}\right) \leqslant f_{\mathcal{G}} - |\mathcal{S}|f_{\mathcal{S}} - |\bar{\mathcal{S}}|f_{\bar{\mathcal{S}}} \leqslant \lim_{r\to\infty}\frac{1}{\beta N_{o,r}}\ln\left(\mathrm{e}^{\beta\sup_{(i,j)}J_{ij}|\partial(\mathcal{S},\bar{\mathcal{S}})|_r}\right) \tag{C17}$$

and then

$$\lim_{r\to\infty} -\beta \sup_{(i,j)} J_{ij} \frac{|\partial(\mathcal{S},\bar{\mathcal{S}})|_r}{N_{o,r}} \leqslant f_{\mathcal{G}} - |\mathcal{S}| f_{\mathcal{S}} - |\bar{\mathcal{S}}| f_{\bar{\mathcal{S}}} \leqslant \lim_{r\to\infty} \beta \sup_{(i,j)} J_{ij} \frac{|\partial(\mathcal{S},\bar{\mathcal{S}})|_r}{N_{o,r}}. \tag{C18}$$

Since the measure of the boundary $\partial(\mathcal{S},\bar{\mathcal{S}})$ is zero we have that $\lim_{r\to\infty} |\partial(\mathcal{S},\bar{\mathcal{S}})|_r/N_{o,r} = 0$ and therefore we obtain $f_{\mathcal{G}} - |\mathcal{S}| f_{\mathcal{S}} - |\bar{\mathcal{S}}| f_{\bar{\mathcal{S}}} = 0$, that is equation (40). Notice that this result is very general and it exploits only the fact that the symmetry group is compact and that the interactions are bounded.

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